Exercise 28

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$x^2 + 2xy + 4y^2 = 12, \quad (2, 1), \quad \text{(ellipse)}$$

Solution

The aim is to evaluate y' at $x = 2$ and $y = 1$ in order to find the slope there. Differentiate both sides of the given equation with respect to x.

$$\frac{d}{dx}(x^2 + 2xy + 4y^2) = \frac{d}{dx}(12)$$

$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + 4\frac{d}{dx}(y^2) = 0$$

$$(2x) + 2\left\{\left[\frac{d}{dx}(x)\right] y + x\left[\frac{d}{dx}(y)\right]\right\} + 4\left[2y \cdot \frac{d}{dx}(y)\right] = 0$$

$$2x - 2[(1)y + x(y')] + 4(2yy') = 0$$

$$2x - 2y - 2xy' + 8yy' = 0$$

Solve for y'.

$$2x - 2y = (2x - 8y)y'$$

$$y' = \frac{2x - 2y}{2x - 8y}$$

$$y' = \frac{x - y}{x - 4y}$$

Evaluate y' at $x = 2$ and $y = 1$.

$$y'(2, 1) = \frac{(2) - (1)}{(2) - 4(1)} = -\frac{1}{2}$$

Therefore, the equation of the tangent line to the curve represented by $x^2 + 2xy + 4y^2 = 12$ at $(2, 1)$ is

$$y - 1 = -\frac{1}{2}(x - 2).$$

Below is a graph of the curve and the tangent line at $(2, 1)$.